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Unsteady heat distribution in an orthotropic rectangular rod moving along the direction of its length*

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In this paper partial differential equation governing heat conduction in a moving orthotropic solid rod with square cross section has been solved. The finite rod is initially kept at a given temperature and temperatures at both ends are supposed to be known. Some examples and their numerical calculations have also been added to indicate applications of the solution.

Introduction

The problems of energy distribution in anisotropic and orthotropic solids are of practical importance. Such problems occur when we consider heat conduction in crystals, rocks, wood and laminated materials such as transformer cores etc.

Carslaw & Jaeger (1959) suggested a few problems of those solids treated under ordinary conditions. But in applied physics we come across many cases in which it is required to determine unsteady temperature distribution in orthotropic solids of different shapes moving in conductive media. No mathematical treatment of such problems has been considered so far.

In this paper we attempt to solve a partial differential equation governing heat conduction in a moving orthotropic solid rod with square cross section having a source situated inside it. Initially the solid is kept at a prescribed temperature and the temperatures of both extreme bases are known.

In one simple case of boundary and initial conditions we have calculated and computed numerical values of the temperature and subsequently discussed its variations with respect to important variable quantities.

Equation of Motion and Boundary Conditions

The partial differential equation of heat condition in a solid moving along the direction of z-axis with a constant velocity U, is given by

$$\frac{\partial}{\partial x}\left[K_x\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[K_y\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial z}\left[K_z\frac{\partial u}{\partial z}\right]-U\frac{\partial u}{\partial z}+Q(x,y,z,t)=c\frac{\partial u}{\partial t} \quad \dots \quad (1)$$

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where K_x, K_y, K_z are conductivities along the directions of the principal axes, $Q(x, y, z, t)$ is intensity of continuous source of heat situated at the point x, y, z.

Consider a rod of length l with a square cross section and let the material be orthotropic in xy plane, assuming,

$$K_x = k(1-x^2), \quad K_y = k(1-y^2), \quad K_z = k$$

$$Q(x, y, z, t) = (c_1+c_2x)\frac{\partial u}{\partial x} + (c_3+c_4y)\frac{\partial u}{\partial y} + au + b + \rho(x, y, z, t); \qquad \dots \quad (2)$$

c_1, c_2, c_3, c_4, a and b are constants and $\rho(x, y, z, t)$ is an arbitrary function of xyz and t.

The mathematical statement of the conditions are given by

$$\begin{aligned} &u(x, y\, 0, t) = \phi(x, y, t),\ t > 0;\ u(x, y, z, 0) = \psi(x, y, z); \\ &u(x, y, l, t) = 0,\ t > 0. \end{aligned} \qquad (3)$$

There is no radiation from the long sides of the rod. The equations of both side ends are given below.

Solution of the Problem

To solve the problem we use the Jacobi transform of two variables as defined by Saxena (In press) :

$$\begin{aligned} J\{h(x, y)\} = \int_{-1}^{1}\int_{-1}^{1} &(1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\gamma}(1+y)^{\delta}P_m^{(\alpha,\beta)}(x) \\ &\times P_n^{(\gamma,\delta)}(y)h(x, y)dxdy \end{aligned} \qquad \dots \quad (4)$$

provided $\alpha, \beta, \gamma, \delta > -1$.

The inversion formula of the above transform gives

$$h(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (\delta_m\delta_n')^{-1}P_m^{(\alpha,\beta)}(x)P_n^{(\gamma,\delta)}(y)J\{h(x, y)\} \qquad \dots \quad (5)$$

where

$$\delta_m = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!(\alpha+\beta+2m+1)\Gamma(m+\alpha+\beta+1)},\ \delta_n{}^1 = \frac{2^{\gamma+\delta+1}\Gamma(n+\gamma+1)\Gamma(n+\delta+1)}{n!(\gamma+\delta+2n+1)\Gamma(n+\gamma+\delta+1)} \qquad (5a)$$

Also, we have the following theorem :

If (i) the function $h(x, y)$ and its partial derivatives $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ are bounded almost continuous in the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$,

(ii) the second derivatives $\frac{\partial^2 h}{\partial x^2}$ and $\frac{\partial^2 h}{\partial y^2}$ are bounded and integrable at each point of the square given in (i)

(iii) the Jacobi transform of two variables of $h(x, y)$ exists,

and

(iv)
$$\lim_{x\to\pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}h(x, y) = \lim_{x\to\pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{\partial h}{\partial x} = 0,$$

$$\lim_{y\to\pm 1}(1-y)^{\gamma+1}(1+y)^{\delta+1}h(x, y) = \lim_{y\to\pm 1}(1-y)^{\gamma+1}(1+y)^{\delta+1}\frac{\partial h}{\partial y} = 0$$

then,

$$J\{h(x, y)\} = -\{m(m+\alpha+\beta+1)+n(n+\gamma+\delta+1)\}J\{h((x, y)\} \quad \ldots \quad (6)$$

where

$$C\{h(x, y)\} = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{\partial}{\partial x}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{\partial h}{\partial x}\right\}$$

$$+(1-y)^{-\gamma}(1+y)^{-\delta}\frac{\partial}{\partial y}\left\{(1-y)^{\gamma+1}(1+y)^{\delta+1}\frac{\partial h}{\partial y}\right\},$$

Now substituting the values of K_x, K_y, K_z and Q from (2) in (1) with the assumptions

$$\alpha = -\frac{c_1+c_2}{2k},\ \beta = -\frac{c_1-c_2}{2k},\ \gamma = -\frac{c_3+c_4}{2k},\ \delta = -\frac{c_3-c_4}{2k},$$

taking Jacobi transform of two variables of the same equation and using (6) we obtain

$$\frac{\partial^2 u_J}{\partial z^2} - \frac{U}{k}\frac{\partial u_J}{\partial z} - Au_J = \frac{c}{k} - B - \rho_J(z, t), \quad \ldots \quad (7)$$

where

$$u_J(z, t) = J\{u(x, y, z, t)\},\ \rho_J(z, t) = J\{\rho(x, y, z, t)\},$$
$$A = m(m+\alpha+\beta+1)+n(n+\gamma+\delta+1)-a/k \text{ and } B = b\delta_m\delta_n'. \quad \ldots \quad (7a)$$

The conditions (3) reduce to

$$u_J(0, t) = \phi_J(t),\ u_J(l, t) = 0 \text{ and } u_J(z, 0) = \psi_J(z) \quad \ldots \quad (8)$$

where

$$\phi_J(t) = J\{\phi(x, y, t)\} \quad \text{and} \quad \psi_J(z) = J\{\psi(x, y, z)\}.$$

Here we use the well known Laplace transform defined as

$$L\{h(t)\} = \int_0^\infty e^{-pt}h(t)dt \quad \ldots \quad (9)$$

and the following inversion formula associated with the transform :

$$h(t) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} e^{pt} L\{h(t)\} dp, \ \nu > 0. \quad \dots (10)$$

Also we have

$$L\left\{ \frac{\partial^n h(t)}{\partial t^n} \right\} = p^n L\{h(t)\} - p^{n-1} h(0) - p^{n-2} h'(0) - \dots - h^{(n-1)}(0). \quad \dots (11)$$

Multiplying (7) by e^{-pt} and integrating with respect to t from 0 to ∞ and using (11) we get

$$\frac{d^2 v}{dz^2} - \frac{U}{k}\frac{dv}{dz} - \left(A + \frac{cp}{k} \right) v = -\frac{c}{k} \psi_J(z) - \sigma(z, p) - \frac{B}{p}, \quad \dots (12)$$

where

$$v(z, p) = L\{u_J(z, t)\},\ \sigma(z, p) = L\{\rho_J(z, t)\}$$

Now, with the help of the conditions

$$v(0, p) = \theta(p),\ v(l, p) = 0 \quad \dots (13)$$

where

$$\theta(p) = L\{\phi_J(t)\},$$

the solution of (12) is obtained as

$$v(z, p) = \frac{\{\theta(p) - \eta(0, p)\} e^{\frac{Uz}{2k}} \sinh\left\{ \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}} (l - z) \right\}}{\sinh\left\{ \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}} l \right\}}$$

$$+ \frac{\eta(l, p) e^{-\frac{Uz}{2k}} \cosh\left\{ \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}} z \right\}}{\sinh\left\{ \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}} l \right\}}$$

$$+ \frac{1}{2} \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{-\frac{1}{2}} \eta(z, p) + B \left(A + \frac{cp}{k} \right)^{-1} \quad \dots (14)$$

where

$$\eta(z, p) = \frac{1}{\lambda_1 - \lambda_2} \left[e^{\lambda_1 z} \int_0^z e^{-\lambda_1 z} \left\{ \frac{c}{k} \psi_J(z) + \sigma(z, p) \right\} dz \right.$$

$$\left. - e^{\lambda_2 z} \int_0^z e^{-\lambda_2 z} \left\{ \frac{c}{k} \psi_J(z) + \sigma(z, p) \right\} dz \right],$$

$$\lambda_1 = \frac{U}{2k} - \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}},\ \lambda_2 = \frac{U}{2k} + \left(\frac{U^2}{4k^2} + A + \frac{cp}{k} \right)^{\frac{1}{2}}.$$

In this way we obtain the value of $v(z\, p)$ in terms of the known quantities. The value of $u(x, y, z, t)$ can be obtained with the help of (10) and consequently using the inversion formula (5).

The value of $v(z, p)$ thus obtained is in a complicated form. Here we calculate its value for a simple case.

Let

$$\psi(x, y, z) = 0, \; \rho(x, y, z, t) = 0, \; \phi(x, y, t) = 100xy$$

In this case we have

$$v(z, p) = D_J e^{\frac{U_z}{2k}} \frac{\sinh\left\{\left(\frac{U^2}{4k^2}+A+\frac{cp}{k}\right)^{\frac{1}{2}}(l-z)\right\}}{p \sinh\left\{\left(\frac{U^2}{4k^2}+A+\frac{cp}{k}\right)^{\frac{1}{2}} l\right\}} + \frac{kB}{c}\left(\frac{kA}{c}+p\right)^{-1} \quad \ldots \quad (15)$$

where

$$D_J = J(100xy)$$

Using the inversion theorem (10) and the well known result

$$L(e^{-\alpha t}) = \frac{1}{p+\alpha}$$

we arrive at

$$u_J(z, t) = D_J e^{\frac{U_z}{2k}} \cdot \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \frac{e^{pt} \sinh\left\{\left(\frac{U^2}{4k^2}+A+\frac{cp}{k}\right)^{\frac{1}{2}}(l-z)\right\}}{p \sinh\left\{\left(\frac{U^2}{4k^2}+A+\frac{cp}{k}\right)^{\frac{1}{2}} l\right\}} dp + \frac{kB}{c} e^{-\frac{KA}{c}t} \quad \ldots \quad (16)$$

The integrand of (16) has simple poles at $p = 0$ and for

$$l\left(\frac{U^2}{4k^2}+A+\frac{cp}{k}\right)^{\frac{1}{2}} = n\pi i, \quad n = 0, 1, 2, \ldots$$

that is

$$p = -\frac{U^2}{4kc} - \frac{kA}{c} - \frac{kn^2\pi^2}{l^2c}, \quad n = 1, 0, 2, \ldots$$

Evaluating the residues at these poles we get

$$u_J(z, t) = D_J e^{\frac{U_z}{2k}} \frac{\sinh\left\{\left(\frac{U^2}{4k^2}+A\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{U^2}{4k^2}+A\right)^{\frac{1}{2}} l\right\}}$$

$$+\frac{2D_J\pi}{l^2}\, e^{\frac{U_z}{2k}} \sum_{r=1}^{\infty} \frac{(-1)^r\, r \sin\{r\pi(l-z)/l\} e^{-\left(\frac{U^2}{4k}+\frac{kr^2\pi^2}{l^2c}\right)t}}{\left(\frac{U^2}{4k^2}+\frac{r^2\pi^2}{l^2}\right)}$$

$$+\frac{kB}{c} e^{-\frac{kA}{c}t} \quad \dots \quad (17)$$

Finally, applying the inversion formula (5) we obtain

$$u(x, y, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\delta_m \delta_n^1)^{-1} P_m^{(\alpha,\beta)}(x) P_n^{(\gamma,\delta)}(y)$$

$$\times \left[D_J e^{\frac{U_z}{2k}} \frac{\sinh\left\{\left(\frac{U^2}{4k^2}+A\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{U^2}{4k^2}+A\right)^{\frac{1}{2}} l\right\}} + \frac{kB}{c} e^{-\frac{kA}{c}t}\right.$$

$$\left. +\frac{2D_J\pi}{l^2}\, e^{\frac{U_z}{2k}} \sum_{r=0}^{\infty} \frac{(-1)^r\, r \sin\{r\pi(l-z)/l\}}{\left(\frac{U^2}{4k^2}+\frac{r^2\pi^2}{l^2}\right)} e^{-\left(\frac{U^2}{4k}+\frac{kr^2\pi^2}{l^2c}\right)t}\right] \quad \dots \quad (18)$$

where δ_m and $\delta^1{}_n$ are given in (5a) and A and B are given in (7a).

Now if t tends to infinity *i.e.*, allowing it to reach steady state conditions and substituting the value of D obtained with the help of (15) the integrals

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta x P_0^{(\alpha,\beta)}(x)dx = 2^{\alpha+\beta+1}(\beta-\alpha)\times\frac{\Gamma(\alpha+1)(\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)} \quad \dots \quad (19)$$

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta x P_1^{(\alpha,\beta)}(x)dx = 2^{\alpha+\beta+2}\frac{\Gamma(\alpha+2)\,\Gamma(\beta+2)}{\Gamma(\alpha+\beta+4)} \quad \dots \quad (20)$$

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta x P_m^{(\alpha,\beta)}(x)dx = 0 \text{ for } m > 1 \quad \dots \quad (21)$$

we arrive at

$$u(x, y, z, t)_{t\to\infty} = \frac{100 e^{\frac{U_z}{2k}}}{(\alpha+\beta+2)(\gamma+\delta+2)}$$

$$\times \left[\frac{(\beta-\alpha)(\delta-\gamma)\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}\right)^{\frac{1}{2}} l\right\}}\right.$$

$$+\frac{(\beta-\alpha)\{(\gamma+\delta+2)y+\gamma-\delta\}\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+\gamma+\delta+2\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{W^2}{uk^2}-\frac{a}{k}+\gamma+\delta+2\right)^{\frac{1}{2}}l\right\}}$$

$$+\frac{(\delta-\gamma)\{(\alpha+\beta+2)x+\alpha-\beta\}\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+\alpha+\beta+2\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+\alpha+\beta+2\right)^{\frac{1}{2}}l\right\}}$$

$$+\frac{\{(\alpha+\beta+2)x+\alpha-\beta\}\{(\gamma+\delta+2)y+\gamma-\delta\}}{\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+\alpha+\beta+\gamma+\delta+4\right)^{\frac{1}{2}}l\right\}}$$

$$\left.\times\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+\alpha+\beta+\gamma+\delta+4\right)^{\frac{1}{2}}(l-z)\right\}\right] \qquad \ldots \quad (22)$$

Further in the expression (2) of the intensity $Q(x, y, z, t)$ we take

$$c_1 = c_3 = k, \quad c_2 = c_4 = 0$$

which gives $$\alpha = \beta = \gamma = \delta = -\tfrac{1}{2}$$

In this case the expression (22) reduces to

$$u(x, y, z) = 100xye^{\frac{Uz}{2k}}\frac{\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+2\right)^{\frac{1}{2}}(l-z)\right\}}{\sinh\left\{\left(\frac{U^2}{4k^2}-\frac{a}{k}+2\right)^{\frac{1}{2}}l\right\}} \qquad \ldots \quad (23)$$

where $$u(x, y, z) = u(x, y, z, t)_{t\to\infty}$$

Here we discuss two different cases depending on the values of U, k and a.

Case 1 : $$a > \frac{U^2}{4k}+2k.$$

This case may arise when a remains constant and the motion of the solid is slow or U is known and k large

For the sake of convenience we take

$$\frac{a}{k} = \frac{U^2}{4k^2}+\frac{7}{4}, \quad l = \pi;$$

so that the expression (23) takes the form

$$u(x, y, z) = 100xye^{\frac{Uz}{2k}}\cos\frac{z}{2} \qquad \ldots \quad (24)$$

The *Isothermal Surfaces* for $u = T$ are given as

$$xye^{\frac{Uz}{2k}} \cos \frac{z}{2} = \frac{T}{100} \quad \text{for} \quad U > 0 \qquad \dots \quad (25)$$

$$xy \cos \frac{z}{2} = \frac{T}{100} \quad \text{for} \quad U = 0 \qquad \dots \quad (26)$$

In the later case the *Isothermal Curves* at the plane $z = \frac{2\pi}{3}$ are shown in figure 1, for $T = 5,10,20,25,30$ and 40. The variation of temperature in this case along any line parallel to the z-axis will be a simple cosine curve.

In the general case ($U > 0$), the variation is shown in figure 2, We have considered the line $x = y = 0.5$ and the cases $U/k = 1$ and $U/k = 2$.

If we give different values to the velocity U we obtain different values of temperature considered at certain fixed points.

TABLE 1. Temperature corresponding to different values of U/k at 0.2, 0.1 and $2\pi/3$.

U/k	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
u	1	1.37	1.87	2.56	3.51	4.81	6.59	9.02	12.34	16.90	23.14

Case 2 :

$$\alpha < \frac{U^2}{4k} + 2k.$$

This case may come for rapid motions of the solid or small values of k. For simplicity we take

$$\frac{U^2}{4k^2} = \frac{a}{k} - 1$$

so that expression (23) takes the form

$$u(x, y, z) = 100xye^{\frac{Uz}{2k}} \sinh\,(l-z)/\sinh l \qquad \dots \quad (27)$$

Isothermal surfaces are of form

$$xye^{\frac{Uz}{2k}} \sinh\,(l-z) = \text{constant}, \quad \text{for} \quad U > 0 \qquad \dots \quad (28)$$

and

$$xy \sinh\,(l-z) = \text{constant}, \quad \text{for} \quad U = 0 \qquad \dots \quad (29)$$

Variations of temperature along the direction of z-axis is shown in figures 3A and 3B.

The examples and numerical calculations for unsteady state distributions and more general conditions may lead to certain interesting results of practical importance. These results will be reported elsewhere.

References

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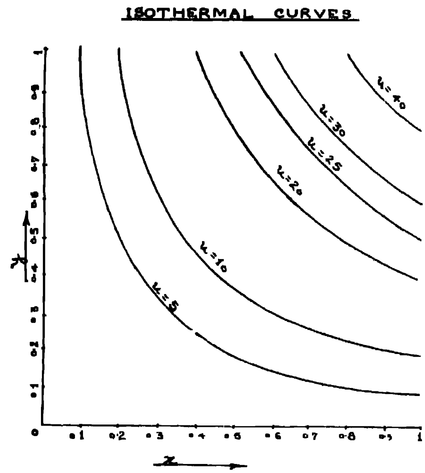

Figure 1. Isothermal curves at the plane $z = \frac{2\pi}{3}$ for $U = 0$.

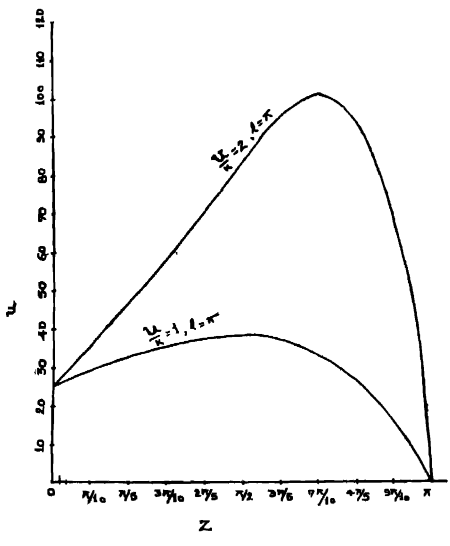

Figure 2. Curves showing variations of u along the line $x = y = 0.5$ for $U/k = 1$ and $U/k = 2$.

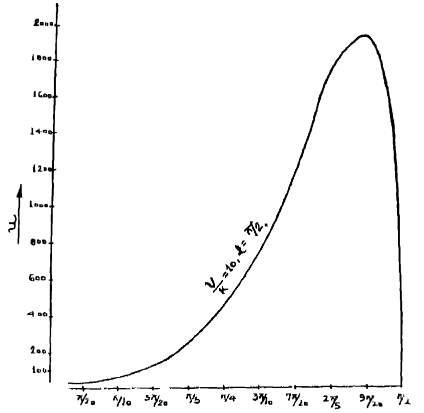

Figure 3A. The variation of temperature along the line $x = y = 0.5$ for $U/k = 10$, $l = \frac{\pi}{2}$.

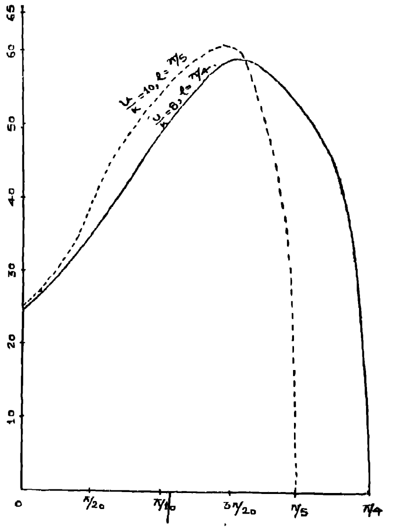

Figure 3B. The variation of temperature along the line $x = y = 0.5$ for $U/k = 10$, $l = \frac{\pi}{5}$ and $U/k = 8$, $l = \frac{\pi}{4}$.